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A Conjecture for the Superintegrable Chiral Potts Model

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Abstract We adapt our previous results for the "partition function" of the superintegrable chiral Potts model with open boundaries to obtain the corresponding matrix elements of $\mathrm{e}^{-\alpha H}$, where H is the associated Hamiltonian. The spontaneous magnetization $\mathcal{M}_r$ can be expressed in terms of particular matrix elements of $\mathrm{e}^{-\alpha H} S_1^r \mathrm{e}^{-\beta H}$, where S_1 is a diagonal matrix. We present a conjecture for these matrix elements as an m by m determinant, where m is proportional to the width of the lattice. The author has previously derived the spontaneous magnetization of the chiral Potts model by analytic means, but hopes that this work will facilitate a more algebraic derivation, similar to that of Yang for the Ising model.

Keywords Statistical mechanics · Lattice models · Transfer matrices

1 Introduction

In a previous paper [1], we obtained the partition function $\tilde{Z}_Q$ (here referred to as $\tilde{Z}_p$) of the superintegrable chiral Potts model with open boundary conditions. It is a simple product of elements of two-by-two matrices, reflecting the fact that there is a reduced representation in which the transfer matrices have a direct product structure, similar to that of the Ising model [2].

Very recently, we have considered the problem of calculating the spontaneous magnetization $\mathcal{M}$ of the square lattice Ising model [3]. We used the method of Yang [4] and defined $\mathcal{M}$ in terms of the partition function $\widetilde{W}$ on a cylindrical lattice of L columns with fixed-spin boundary conditions on the upper and lower rows, with a single-spin operator S_1 acting on a spin located within the lattice. For convenience, we took the limit when the transfer matrix could be replaced by the exponential of an associated Hamiltonian.

The Clifford algebra technique of Kaufman [5] can still be applied to this system, so that $\widetilde{W}$ can be calculated as the square root of an L-dimensional determinant. This can be further reduced to a determinant (without the square root) of dimension approximately $L/2$.

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Here we write down corresponding definitions of $\widetilde{W}$ for the N-state superintegrable chiral Potts model, which reduces to the Ising model when $N = 2$. We conjecture in (7.1)–(7.3) that $\widetilde{W}$ is also given by a determinant of dimension smaller than L, being a fairly immediate generalization of that for the Ising case. If true, this is an exact formula for finite lattices, containing three additional arbitrary parameters α, β, x in addition to N, L and the labels p, q of the appropriate sub-spaces. It is therefore easy to test numerically, and we have tested it to 60 or more digits of accuracy for various small values of N, L (up to $N + L = 10$).

If this conjecture is indeed true, then the spontaneous magnetization of the superintegrable chiral Potts model is given by the expression (7.11) below. This necessitates taking the limit $L \to \infty$. As yet we have not done this, but we have observed numerically that (7.11) does indeed appear to converge to the known result (5.5). The author has previously derived (7.11) by analytic methods [6, 7] that apply in the large-lattice limit, but it would still be interesting to have an algebraic derivation that could give greater insight into the properties of the model on a finite lattice.

2 Partition Function

2.1 Definition

The model is defined on the square lattice, rotated through 45°, with $M + 1$ horizontal rows, each containing L spins, as in Fig. 1.

We impose cylindrical boundary conditions, so that the last column L is followed by the first column 1. At each site i there is a spin σ_i, taking the values $0, 1, \ldots, N-1$. The spins in the bottom row are fixed to have value a, those in the top row to have value 0. Adjacent spins σ_i, σ_j on southwest to northeast edges (with i below j) interact with Boltzmann weight $\mathcal{W}(\sigma_i - \sigma_j)$; those on southeast to northwest edges with weight $\overline{\mathcal{W}}(\sigma_i - \sigma_j)$.

These $\mathcal{W}, \overline{\mathcal{W}}$ are the Boltzmann weight functions:

$$\begin{aligned} \mathcal{W}(n) &= \mathcal{W}(n+N) = \mu^n \prod_{j=1}^{n} (1 - \omega^j y)/(1 - \omega^j x), \\ \overline{\mathcal{W}}(n) &= \overline{\mathcal{W}}(n+N) = \mu^{-n} \prod_{j=1}^{n} (\omega - \omega^j x)/(1 - \omega^j y), \end{aligned} \tag{2.1}$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/n}$, x and y are complex parameters, and

$$\mu^N = (x^N - 1)/(y^N - 1). \tag{2.2}$$

An important associated parameter is

$$k' = (x^N - 1)(y^N - 1)/(y^N - x^N). \tag{2.3}$$

The partition function, which depends on a, is

$$Z_a = \sum_{\sigma} \prod_{\langle i,j \rangle} \mathcal{W}(\sigma_i - \sigma_j) \prod_{\langle i,j \rangle} \overline{\mathcal{W}}(\sigma_i - \sigma_j), \tag{2.4}$$

the products being over all edges of the two types. The sum is over all values of all the free spins. The partition function can be written as

$$Z_a = u_a^{\dagger} T^M u_0, \tag{2.5}$$

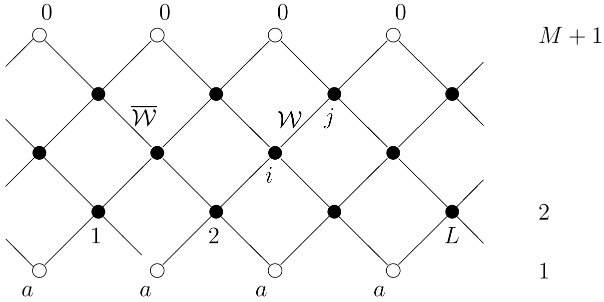

Fig. 1 The square lattice $\mathcal{L}$ turned through 45°

where T is the row-to-row transfer matrix, with elements

$$T_{\sigma,\sigma'} = \prod_{i=1}^{L} \mathcal{W}(\sigma_i - \sigma'_{i+1})\overline{\mathcal{W}}(\sigma_i - \sigma'_i), \tag{2.6}$$

σ being the set of all spins $\sigma_1, \ldots, \sigma_L$ in one row, and σ' being the set in the row above. Thus T is an N^L by N^L matrix. The vector u_a is of dimension N^L, with entries

$$(u_a)_\sigma = \begin{cases} 1 & \text{if } \sigma_1 = \cdots = \sigma_L = a, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

The superintegrable chiral Potts model is a special case of the more general solvable chiral Potts model, which satisfies the star-triangle relation [8]. This ensures that two transfer matrices T, T', with different values of x, y, but the same value of k', commute.

2.2 The Spin-Increment Matrix R

Let R be the N^L by N^L matrix with entries

$$R_{\sigma,\sigma'} = \prod_{j=1}^{L} \delta(\sigma_j, \sigma'_j + 1), \tag{2.8}$$

where $\delta(a,b) = 1$ if $a = b$ (modulo N), else $\delta(a,b) = 0$. Then pre-multiplying by R has the effect of increasing all spins by 1 (modulo N), hence $Ru_a = u_{a+1}$ and R commutes with T:

$$RT = TR. \tag{2.9}$$

For this reason it is natural to use the Fourier transform of u_a:

$$v_p = N^{-1/2} \sum_{a=0}^{N-1} \omega^{-ap} u_a. \tag{2.10}$$

taking $p = 0, \ldots, N-1$. This p replaces the Q of [1, 9]. Then

$$Rv_p = \omega^p v_p. \tag{2.11}$$

If we also define

$$\tilde{Z}_p = \sum_{a=0}^{N-1} \omega^{pa} Z_a, \tag{2.12}$$

then

$$\tilde{Z}_p = N^{1/2} v_p^\dagger T^M u_0$$
$$= \sum_{q=0}^{N-1} v_p^\dagger T^M v_q. \tag{2.13}$$

From (2.9), we can replace T^M in the summand by $R^{-1}T^M R$, and from (2.11) this is equivalent to multiplying by ω^{q-p}. This in turn means the summand must vanish unless $q = p$, so

$$\tilde{Z}_p = v_p^\dagger T^M v_p. \tag{2.14}$$

2.3 The Sub-Space V_p

Following the observations of Albertini *et al.* [10], we showed in [1, 9] that if one operates on v_p by any product of matrices T, with different values of x, y but the same value of k', then all the vectors generated lie in a vector space V_p, where $p = 0, \ldots, N-1$. For any vector v in V_p,

$$Rv = \omega^p v. \tag{2.15}$$

We also showed that the transfer matrices satisfied a functional relation that determined their eigenvalues, and derived the result (2.24) for the partition function $\tilde{Z}_p$.

If

$$m = m_p = \text{integer part of } \left[\frac{(N-1)L - p}{N}\right], \tag{2.16}$$

then there are just 2^m distinct eigenvalues. What we have not shown, but believe to be true, is that each such eigenvalue occurs just once, so that V_p is of dimension 2^m. Certainly, by continuity from the case $k' = 0$, the largest eigenvalue (which is the one we most often consider) occurs just once.

For the case when $p = 0$ and L divides by N, Au-Yang and Perk have recently obtained the eigenvectors explicitly [11].

Two vectors v, w in different spaces V_p, V_p (with $q \neq p$) are necessarily orthogonal, i.e. $v^\dagger \cdot w = 0$.

Define

$$P(z^N) = z^{-p} \sum_{n=0}^{N-1} \omega^{(L+p)n} (z^N - 1)/(z - \omega^n)^L. \tag{2.17}$$

Then $P(w) = P_p(w)$ is a polynomial in w of degree m. Let its zeros be $w_1, \ldots, w_m$ and define $\theta_1, \ldots, \theta_m = \theta(p, 1), \ldots, \theta(p, m)$ by

$$\cos\theta_j = \cos[\theta(p, j)] = (1 + w_j)/(1 - w_j), \quad 0 < \theta_i < \pi, \tag{2.18}$$

for $j = 1, \ldots, m$. Set

$$G = (x^N y^N - 1)/(y^N - x^N), \tag{2.19}$$
$$g = N(1 - x^{-1})/(1 - x^{-N}), \tag{2.20}$$

define the two-by-two matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.21}$$

$$F(x, y, \theta) = \frac{1 - x^{-N}}{2k'} \left[G I_2 + (1 - k' \cos\theta) S - k' \sin\theta C \right], \tag{2.22}$$

I_2 being the identity matrix, and set

$$D(\cos\theta) = (1 \quad 0) \cdot F(x, y, \theta)^M \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.23}$$

then in [1] we find that

$$\tilde{Z}_p = g^{LM} x^{-Mp} D(\cos\theta_1) D(\cos\theta_2) \cdots D(\cos\theta_m). \tag{2.24}$$

3 The Hamiltonian Limit

Take

$$\mu = \mathrm{e}^{-2\epsilon} \tag{3.1}$$

and consider the limit when $\epsilon \to 0$. Then to first order in ϵ

$$x = y = 1 + 2k'\epsilon, \tag{3.2}$$

$$\mathcal{W}(n) = 1 - 2n\epsilon, \qquad \overline{\mathcal{W}}(n) = 2k'\epsilon/(1 - \omega^{-n}) \tag{3.3}$$

for $0 < n < N$, while $\mathcal{W}(0) = \overline{\mathcal{W}}(0) = 1$. Noting that

$$N - 1 - 2j = 2 \sum_{n=1}^{N-1} \frac{\omega^{nj}}{1 - \omega^{-n}} \tag{3.4}$$

for $0 \le j < N$, it follows that

$$T = [1 - (N-1)L\epsilon] I - \epsilon \mathcal{H}, \tag{3.5}$$

where I is the identity matrix and

$$\mathcal{H} = -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} (\mathcal{Z}_j^n \mathcal{Z}_{j+1}^{-n} + k' X_j^n)/(1 - \omega^{-n}). \tag{3.6}$$

This is the Hamiltonian associated with the transfer matrix T. Since all transfer matrices with the same value of k' commute, they also commute with $\mathcal{H}$. Here $\mathcal{Z}_j, X_j$ are the N^L by N^L matrices of [10], with elements

$$\left(\mathcal{Z}_j\right)_{\sigma,\sigma'} = \omega^{\sigma_j} \prod_{m=1}^{L} \delta(\sigma_m, \sigma'_m), \tag{3.7}$$

$$\left(X_j\right)_{\sigma,\sigma'} = \delta(\sigma_j, \sigma'_j + 1) \prod_{n=1}^{L}{}^{*} \delta(\sigma_n, \sigma'_n), \tag{3.8}$$

the $*$ on the last product indicating that it excludes the case $n = j$.

The Hamiltonian $\mathcal{H}$ is known to have very special properties. In particular Au-Yang and Perk showed that it satisfies the "Onsager algebra" [12].

Still working to first order in ϵ, we obtain

$$g = 1 + (N-1)k'\epsilon,$$
$$\frac{(1-x^{-N})G}{2k'} = 1 - N(1+k')\epsilon,$$
$$F(x,y,\theta) = [1 - N(1+k')\epsilon]I_2 + N\epsilon\left[(1-k'\cos\theta)S - k'\sin\theta C\right].$$

Now we take

$$\epsilon = \alpha/M, \tag{3.9}$$

and let $M \to \infty$, keeping α fixed. Then

$$F(x,y,\theta)^M \to \exp\{N\alpha[-(1+k')I_2 + (1-k'\cos\theta)S - k'\sin\theta C]\} \tag{3.10}$$

and from (3.5),

$$T^M \to \mathrm{e}^{-(N-1)L\alpha}\exp(-\alpha\mathcal{H}). \tag{3.11}$$

From (2.23) and (2.24), it follows that

$$v_p^\dagger \exp(-\alpha\mathcal{H})v_p = e^{-\mu\alpha}\overline{D}(\cos\theta_1)\cdots\overline{D}(\cos\theta_m), \tag{3.12}$$

where

$$\mu = \mu_p = 2k'p + (1+k')(mN - NL + L), \tag{3.13}$$

$$\overline{D}(\cos\theta) = (1 \quad 0)\cdot \exp[-\alpha\tilde{F}(\theta)]\cdot \begin{pmatrix}1\\0\end{pmatrix} \tag{3.14}$$

and $\tilde{F}(\theta)$ is the two-by-two matrix

$$\tilde{F}(\theta) = -N(1-k'\cos\theta)S + Nk'\sin\theta C. \tag{3.15}$$

3.1 The Two-by-Two Exponential

We can calculate the exponential in (3.14) of the two-by-two matrix $-\alpha\tilde{F}(\theta)$ in the obvious way, by diagonalizing it, exponentiating, and then returning to the original basis. If we define

$$\begin{aligned}\lambda = \lambda(\theta) &= (1 - 2k'\cos\theta + k'^2)^{1/2},\\ u_p(\alpha,\theta) &= \cosh(N\alpha\lambda) + \frac{1-k'\cos\theta}{\lambda}\sinh(N\alpha\lambda)\end{aligned} \tag{3.16}$$

$$\begin{aligned}v_p(\alpha,\theta) &= -\frac{k'\sin\theta}{\lambda}\sinh(N\alpha\lambda)\\ w_p(\alpha,\theta) &= \cosh(N\alpha\lambda) - \frac{1-k'\cos\theta}{\lambda}\sinh(N\alpha\lambda),\end{aligned} \tag{3.17}$$

then

$$\exp[-\alpha \tilde{F}(\theta)] = \begin{pmatrix} u_p(\alpha,\theta) & v_p(\alpha,\theta) \\ v_p(\alpha,\theta) & w_p(\alpha,\theta) \end{pmatrix}. \tag{3.18}$$

Hence

$$\overline{D}(\cos\theta) = u_p(\alpha,\theta) \tag{3.19}$$

and (3.12) becomes

$$v_p^{\dagger} \exp(-\alpha\mathcal{H}) v_p = e^{-\mu_p \alpha} u_p(\alpha,\theta_1)\cdots u_p(\alpha,\theta_m). \tag{3.20}$$

4 Reduced Representation of $\mathcal{H}$

We consider some basis of the 2^m-dimensional vector space V_p and label the vectors by $s = \{s_1,\ldots,s_m\}$, where each s_i takes the values 1 or -1. We can think of the s_i as "Ising spins". Thus there are 2^m vectors $v_s = v_s^p = v(s_1,\ldots,v_m)$, each of dimension N^L.

In [1] we showed that we can choose the vectors v_s so that v_p above is

$$v_p = v(1,1,\ldots,1), \tag{4.1}$$

and

$$\mathcal{H} v(s_1,\ldots,s_m) = \left[\mu - N\sum_{j=1}^{m}(1-k'\cos\theta_j)s_j\right] v(s_1,\ldots,s_m)$$
$$+ Nk'\sum_{j=1}^{m}\sin\theta_j\, v(s_1,\ldots,-s_j,\ldots,s_m). \tag{4.2}$$

Defining 2^m by 2^m matrices S_j, C_j by

$$(S_j)_{s,s'} = s_j \prod_{n=1}^{m} \delta(s_n, s'_n), \tag{4.3}$$

$$(C_j)_{s,s'} = \delta(s_j, -s'_j) \prod_{n=1}^{m}{}^{*} \delta(s_n, s'_n), \tag{4.4}$$

where again the $*$ means that the term $n = j$ is excluded from the product, we see that with respect to this basis the Hamiltonian $\mathcal{H}$ is now

$$H = \mu_p - N\sum_{j=1}^{m}[(1-k'\cos\theta_j)S_j - k'\sin\theta_j C_j], \tag{4.5}$$

which is (2.20) of [1]. This is consistent with our result (3.12) above.

From (3.13), (4.5), H is linear in k'. Set

$$H = H_0 + k' H_1, \tag{4.6}$$

H_0, H_1 being independent of k', and define

$$\kappa(s) = \sum_{j=1}^{m} (1-s_j)/2, \tag{4.7}$$

then $\kappa(s)$ takes the integer values $0, 1, \ldots, m$. If we order the rows and columns of H with increasing values of $\kappa(s)$, then H_0 is diagonal and H_1 is block tri-diagonal, with non-zero entries only when $|\kappa(s) - \kappa(s')| \leq 1$.

From (4.5), H is a direct sum of m two-by-two matrices. Similarly, if we define the two-by-two matrix

$$U_j = \begin{pmatrix} u_p(\alpha, \theta_j) & v_p(\alpha, \theta_j) \\ v_p(\alpha, \theta_j) & w_p(\alpha, \theta_j) \end{pmatrix}, \tag{4.8}$$

then $\exp(-\alpha H)$ is the direct product

$$\exp(-\alpha H) = e^{-r\alpha} U_1 \otimes U_2 \otimes \cdots \otimes U_m. \tag{4.9}$$

Let $|0\rangle$ be the 2^m-dimensional vector whose elements s are zero except for the element $s_1 = 1, s_2 = 1, \ldots, s_m = 1$, which is unity, i.e.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.10}$$

This is the representative of the N^L-dimensional vector v_p. If $\langle 0|$ is the transpose of $|0\rangle$, then

$$v_p^{\dagger} \mathrm{e}^{-\alpha \mathcal{H}} v_p = \langle 0| \mathrm{e}^{-\alpha H} |0\rangle \tag{4.11}$$

and (3.20) follows immediately.

The derivation of [1, 9] does not exclude the possibility that the basis vectors v_s depend on the parameter k'. However, all studies for small N, L agree with the hypothesis that they are (or at least can be chosen to be) *independent* of k'. This is consistent with the fact that both $\mathcal{H}$ and H are linear in k'.

5 The Spontaneous Magnetization

Consider the lattice of Fig. 1 and take $a = 0$, so all upper and lower boundary spins are fixed to be zero. Let ζ be the spin on a site deep inside the lattice. Then in the usual way we can define the order parameters of the chiral Potts model as

$$\mathcal{M}_r = \langle \omega^{r\zeta} \rangle \tag{5.1}$$

for $r = 1, \ldots, N-1$. Here $\langle f(\zeta) \rangle$ denotes the usual statistical mechanical average

$$\langle f(\zeta) \rangle = Z_0^{-1} \sum_{\sigma} f(\zeta) \prod_{\langle i,j \rangle} \mathcal{W}(\sigma_i - \sigma_j) \prod_{\langle i,j \rangle} \overline{\mathcal{W}}(\sigma_i - \sigma_j) \tag{5.2}$$

for any function f. We take the limit when the lattice is infinitely large, so $L, M \to \infty$, and ζ is infinitely far from the boundaries.

The $\mathcal{W}, \overline{\mathcal{W}}$ products are unchanged by incrementing all spins by one, so if we imposed toroidal boundary conditions, then it would be true that

$$\langle f(\zeta+1)\rangle = \langle f(\zeta)\rangle \tag{5.3}$$

and this would imply that $\mathcal{M}_r = \omega^r \mathcal{M}_r$. Hence for $r \neq 0$ (mod N) we would necessarily have $\mathcal{M}_r = 0$.

At high temperatures ($k' \geq 1$), this is true also for our fixed-spin boundary conditions when we take the large-lattice limit. However, at lower temperatures ($0 < k' < 1$) the system has ferromagnetic long-range order and "remembers" the boundary conditions even in the limit of ζ deep inside a large lattice, and

$$0 < \mathcal{M}_r < 1. \tag{5.4}$$

In fact we know $\mathcal{M}_r$. In 1989 Albertini *et al.* [10] conjectured that

$$\mathcal{M}_r = (1-k'^2)^{r(N-r)/2N^2} \tag{5.5}$$

and the author was able to derive this formula in 2005 [6, 7]. The method used was analytic, depending on the star-triangle relation, functional relations and analyticity properties.

When $N = 2$ the chiral Potts model (both superintegrable and general) reduces to the Ising model, whose partition function was obtained by Onsager in 1944 [13]. Onsager announced at a conference in Florence in 1949 that he and Kaufman had solved the spontaneous magnetization and obtained $\mathcal{M}_1 = (1-k'^2)^{1/8}$ [14], but the first published derivation of that result was given by Yang in 1952 [4].

Onsager and Yang's methods were much more algebraic, determining the eigenvalues of the transfer matrix T, and certain elements of the eigenvectors. It would be interesting to obtain a derivation of $\mathcal{M}_r$ that parallels Yang's. The object of this paper is to suggest how one may make progress in that direction.

We introduce the N^L by N^L diagonal matrix $\mathcal{S}_r$ with elements

$$(\mathcal{S}_r)_{\sigma,\sigma'} = \omega^{r\sigma_1} \prod_{j=1}^{L} \delta(\sigma_j, \sigma'_j). \tag{5.6}$$

Note that, for all integers p and r,

$$\mathcal{S}_r v_{p+r} = v_p, \qquad {v_p}^{\dagger} \mathcal{S}_r = v^{\dagger}_{p+r}. \tag{5.7}$$

Because of the cylindrical boundary conditions, we can take the spin ζ to be in any column, so we choose it to be in column 1. Then (5.1) can be written

$$\mathcal{M}_r = W/Z_0, \tag{5.8}$$

where

$$W = u_0^{\dagger} T^j \mathcal{S}_r T^{M-j} u_0, \qquad Z_0 = u_0^{\dagger} T^M u_0, \tag{5.9}$$

j being the number of rows below ζ.

From (2.10) and (2.12),

$$W = N^{-1} \sum_{p,q=0}^{N-1} v_p^{\dagger} T^j \mathcal{S}_r T^{M-j} v_q, \qquad Z_0 = N^{-1} \sum_{p=0}^{N-1} v_p^{\dagger} T^M v_p. \tag{5.10}$$

Since R commutes with T and

$$R\mathcal{S}_r = \omega^{-r}\mathcal{S}_r R, \tag{5.11}$$

it follows from (2.15) that the first summand in (5.10) vanishes unless $q = p + r$, so

$$W = N^{-1}\sum_{p=0}^{N-1} v_p^\dagger T^j \mathcal{S}_r T^{M-j} v_{p+r}, \tag{5.12}$$

interpreting $p + r$ as $p + r$ to modulo N.

For $0 < k' < 1$ and L is large, the N largest eigenvalues of T are asymptotically degenerate, their ratios being of the form $1 + O(\mathrm{e}^{-L\nu})$, ν being a measure of the interfacial tension. However, there is one and only one of these eigenvalues in each of the vector spaces V_p, for $p = 0, \ldots, N-1$.

Since T and $\mathcal{H}$ commute and $\mathcal{H}$ is hermitian, the eigenvectors ψ_p corresponding to these eigenvalues are unitary, so

$$\psi_p^\dagger \psi_q = \delta_{p,q}. \tag{5.13}$$

5.1 Asymptotic Degeneracy

In each sub-space V_p there is single largest eigenvalue Λ_p of the transfer matrix T and these eigenvalues are *asymptotically degenerate*, in the sense that for large L there is a common value Λ such that

$$\Lambda^{-1}\Lambda_p = 1 + O(\mathrm{e}^{-Ls_p}), \tag{5.14}$$

i.e. the ratios of the Λ_p approach unity exponentially rapidly.

This can be seen by considering the series expansion of the eigenvector ψ_q in powers of k'. Since T, $\mathcal{H}$ commute, we can look at the eigenvectors of $\mathcal{H}$, corresponding to the most negative (ground state) eigenvalue.

When $k' = 0$, $\mathcal{H} = \mathcal{H}_0$, where

$$\mathcal{H}_0 = -2\sum_{j=1}^{L}\sum_{n=1}^{N-1} \mathcal{Z}_j^n \mathcal{Z}_{j+1}^{-n}/(1-\omega^{-n}). \tag{5.15}$$

This is diagonal, with minimum eigenvalue $-2L$, when all the L spins are equal. Thus from (2.7), $u_0, \ldots, u_{N-1}$ are ground state eigenvectors.

We can start from one of these eigenvectors and use standard linear perturbation theory to develop a series expansion for the eigenvector of $\mathcal{H}$, starting from the initial eigenvector u_a. This entails changing successively more of the spins from value a to some other value. It will work until all of the spins are changed, when for the first time we come to another of the eigenvectors of $\mathcal{H}_0$. At that stage, and only at that stage, one would have to resolve the degeneracy of the initial eigenvalues. This means that naive perturbation theory works to order k'^L. The calculation only depends on a in so far as it involves the differences (mod N) of the L spins from a. Thus to this order the eigenvalue is independent of the initial choice of a. This is true also of the eigenvalues of T, so $\Lambda_p = \Lambda$, Λ being the common eigenvalue, in agreement with (5.14).

Also, if ψ'_a is this near-eigenvector, then

$$\psi_a'^\dagger u_b = \xi\delta_{a,b}, \tag{5.16}$$

where ξ is independent of a and b, and to this order the actual eigenvectors are

$$\psi_p = N^{-1/2} \sum_{a=0}^{N-1} \omega^{-ap} \psi'_a \tag{5.17}$$

it follows that, for all p,

$$\psi_p^\dagger v_p = \xi. \tag{5.18}$$

In the limit of $j, M-j, L$ large we can replace T^j in (5.10), (5.12) by $\psi_p \Lambda^j \psi_p^\dagger$ (with the appropriate value of p), and T^{M-j} by $\psi_p \Lambda^{M-j} \psi_p^\dagger$, giving

$$v_p^\dagger T^j \mathcal{S}_r T^{M-j} v_{p+r} = \mathrm{e}^{M\Lambda} \xi^* \xi \psi_p^\dagger \mathcal{S}_r \psi_{p+r}, \qquad v_p^\dagger T^M v_p = \mathrm{e}^{M\Lambda} \xi^* \xi \tag{5.19}$$

ξ^* being the complex conjugate of ξ and

$$\psi_p^\dagger \mathcal{S}_r \psi_{p+r} = \text{independent of } p. \tag{5.20}$$

Thus W, Z_0 are the two expressions in (5.19), respectively, and

$$\mathcal{M}_r = \psi_p^\dagger \mathcal{S}_r \psi_{p+r}. \tag{5.21}$$

5.2 Expressions in Terms of $\mathcal{H}$

Rather than continue to work with the transfer matrix T, we find it convenient to instead use the negative exponential of the Hamiltonian and to replace T^j, T^{M-j} in (5.12) by $\mathrm{e}^{-\alpha\mathcal{H}}$, $\mathrm{e}^{-\beta\mathcal{H}}$, and T^M in (5.10) by $\mathrm{e}^{-\alpha\mathcal{H}}$ (with a different α), making them

$$W = N^{-1} \sum_{p=0}^{N-1} \tilde{W}_{p,q}, \qquad Z_0 = N^{-1} \sum_{q=0}^{N-1} \tilde{Z}_q, \tag{5.22}$$

where now, setting $q = p + r$,

$$\widetilde{W}_{p,q} = \widetilde{W}_{p,q}(\alpha, \beta, x) = v_p^\dagger \mathrm{e}^{-\alpha\mathcal{H}} \mathrm{e}^{-\rho\mathcal{J}} \mathcal{S}_r \mathrm{e}^{-\beta\mathcal{H}} v_q \tag{5.23}$$

$$\tilde{Z}_p = \tilde{Z}_p(\alpha) = v_p^\dagger \mathrm{e}^{-\alpha\mathcal{H}} v_p \tag{5.24}$$

and

$$x = \mathrm{e}^{-2N\rho}. \tag{5.25}$$

We have introduced the matrix factor $\mathrm{e}^{-\rho\mathcal{J}}$ immediately pre-multiplying $\mathcal{S}_r$ in (5.23). Here

$$\mathcal{J} = \mathcal{H}_0 + L(N-1)I \tag{5.26}$$

is a diagonal matrix whose entries are $0, 2N, 4N, \ldots, 2N[(N-1)L/N]$. Hence $\widetilde{W}_{p,q}(\alpha, \beta, x)$ is a polynomial in x of degree $[(N-1)L/N]$. This naturally manifests itself in the following working and provides a useful check against errors.

We can think of these $\tilde{Z}_p$, $\widetilde{W}_{p,q}$ as Hamiltonian partition functions. They are rather simpler than the original partition functions to work with.

When $\rho \to +\infty$, then $x \to 0$ and $\mathrm{e}^{-\rho \mathcal{J}} \to v_p {v_p}^\dagger$, so, using (5.7),

$$\begin{aligned} \widetilde{W}_{p,q}(\alpha, \beta, 0) &= v_p^\dagger \mathrm{e}^{-\alpha\mathcal{H}} v_p {v_q}^\dagger \mathrm{e}^{-\beta\mathcal{H}} v_q \\ &= \tilde{Z}_p(\alpha)\tilde{Z}_q(\beta), \end{aligned} \tag{5.27}$$

$$\widetilde{W}_{p,q}(\alpha, 0, x) = \tilde{Z}_p(\alpha), \qquad \widetilde{W}_{p,q}(0, \beta, x) = \tilde{Z}_q(\beta). \tag{5.28}$$

These relations also provide useful checks on our subsequent calculations.

Because $\mathcal{H}, T$ commute, they have the same ground-state eigenvectors ψ_p. In the limit when $\rho = 0$, and $\alpha, \beta, L \to \infty$, we obtain

$$\begin{aligned} \widetilde{W}_{p,q}(\alpha, \beta, 1) &= \mathrm{e}^{-(\alpha+\beta)\Lambda} \xi^* \xi \psi_p^\dagger \mathcal{S}_r \psi_q, \\ \tilde{Z}_p(\alpha) &= \mathrm{e}^{-\alpha\Lambda} \xi^* \xi. \end{aligned} \tag{5.29}$$

So from (5.20), (5.21),

$$\mathcal{M}_r = \lim_{\alpha,\beta,L\to\infty} \frac{\widetilde{W}_{p,q}(\alpha, \beta, 1)}{(\tilde{Z}_p(2\alpha)\tilde{Z}_q(2\beta))^{1/2}} \tag{5.30}$$

for any p, q such that $0 \leq p, q < N$ and $q = p + r$, mod N.

From (3.20) and (5.24),

$$\tilde{Z}_p(\alpha) = e^{-\mu_p \alpha} u_p(\alpha, \theta_1) \cdots u_p(\alpha, \theta_m). \tag{5.31}$$

It remains to calculate $\widetilde{W}_{p,q}(\alpha, \beta, x)$. We have not done this, but the rest of this paper is concerned with presenting a conjecture for it as a determinant of dimension not greater than $(N-1)L/N$. This expression agrees with the known $N = 2$ result for the Ising model, and indeed is a fairly immediate generalization of that result. It has the properties (5.27), (5.28), and has been extensively tested numerically for small values of N, L.

5.3 Expressions in Terms of H

First we remark that if $v \in V_{p+r}$ and $v' = \mathcal{S}_r v$, then from (2.15), $Rv' = \omega^p v'$, so v' is a candidate for the sub-space V_p. However, in general it does not lie within this sub-space. Even so, we can define a matrix $\mathcal{S}^r_{\mathrm{red}}$ of dimension m_p by m_{p+r} by

$$\left(\mathcal{S}^r_{\mathrm{red}}\right)_{s,s'} = \left(v_s^p\right)^\dagger S_r v_{s'}^{p+r}. \tag{5.32}$$

These elements depend on N, L, p, r. They are of course independent of α and β. From our remarks at the end of Sect. 4 that we expect the v_s to be independent of k', the same must be true of the elements of $\mathcal{S}^r_{\mathrm{red}}$.

We can then write (5.23) as

$$\begin{aligned} \widetilde{W}_{p,q} &= \langle 0|\mathrm{e}^{-\alpha H} \mathrm{e}^{-\rho J} \mathcal{S}^r_{\mathrm{red}} \mathrm{e}^{-\beta H'}|0\rangle, \\ \tilde{Z}_p &= \langle 0|\mathrm{e}^{-\alpha H}|0\rangle, \end{aligned} \tag{5.33}$$

where H' is the H of (4.5), (4.6) but with p replaced by $p + r$, and

$$J = H_0 + L(N-1)I = N \sum_{j=1}^{m} (I - S_j) \tag{5.34}$$

is the diagonal matrix with elements $2N\kappa(s)$ in position (s,s).

Let

$$\tilde{u}_p(1,\alpha,\theta)=u_p(\alpha,\theta),\qquad \tilde{u}_p(-1,\alpha,\theta)=v_p(\alpha,\theta) \tag{5.35}$$

and set

$$q=p+r,\qquad m'=m_q,\qquad \mu'=\mu_q,\qquad \theta'_j=\theta(q,j). \tag{5.36}$$

Then we can write these equations more explicitly as

$$\begin{aligned}\widetilde{W}_{p,q}(\alpha,\beta,x)=e^{-\alpha\mu-\beta\mu'}\sum_{s,s'}\tilde{u}_p(s_1,\alpha,\theta_1)\cdots\tilde{u}_p(s_m,\alpha,\theta_m)\\ \times x^{\kappa(s)}\left(\mathcal{S}^r_{\rm red}\right)_{s,s'}\tilde{u}_q(s'_1,\beta,\theta'_1)\cdots\tilde{u}_q(s'_m,\beta,\theta'_{m'}),\end{aligned} \tag{5.37}$$

and

$$\tilde{Z}_p(\alpha)=e^{-\alpha\mu}u_p(\alpha,\theta_1)\cdots u_p(\alpha,\theta_m). \tag{5.38}$$

The non-zero elements (s,s') of the 2^m by $2^{m'}$ matrix $\mathcal{S}^r_{\rm red}$ satisfy $\kappa(s)=\kappa(s')$. If we also order the rows and columns of $\mathcal{S}^r_{\rm red}$ in increasing value of $\kappa(s)$, then this matrix is block-diagonal.

We do not have a direct derivation of $\mathcal{S}^r_{\rm red}$, though of course it can be calculated numerically for small values of N, L from (5.32). In principle it can be calculated from our conjecture (7.2) below. If $\mathbf{s}$ is the m by m' diagonal blocks of $\mathcal{S}^r_{\rm red}$ in the block $\kappa(s)=\kappa(s')=1$, $\mathbf{h}$ is the corresponding m by m block of H_1, and $\mathbf{h}'$ the m' by m' block of H'_1, then this conjecture implies that the double commutator $\mathbf{h}\cdot\mathbf{h}\cdot\mathbf{s}-\mathbf{2h}\cdot\mathbf{s}\cdot\mathbf{h}'+\mathbf{s}\cdot\mathbf{h}'\cdot\mathbf{h}'$ is of rank one. This was a key initial encouraging observation in our search for the expression (7.2).

6 The Orthogonal Matrix *B*

Before stating our conjecture, we define an m by m' real orthogonal matrix $B=B_{pq}$ whose elements involve the $\theta_1,\ldots,\theta_m$ defined by (2.17), (2.18), as well as the $\theta'_1,\ldots,\theta'_{m'}$ defined similarly, but with p replaced by q and m by m'. We must have $p\neq q$.

We define $B=B_{p,q}$ to be the matrix with elements

$$B_{i,j}=f(p,q,i)f(q,p,j)/(\cos\theta_i-\cos\theta'_j), \tag{6.1}$$

where we choose the functions $f(p,q,i)$, $f(q,p,j)$ to ensure that

$$B^TB=I\quad\text{if } m\geq m',\qquad BB^T=I\quad\text{if } m\leq m', \tag{6.2}$$

I again being the identity matrix, of dimension $\min(m,m')$.

6.1 The Case $p<q$

From (2.16), if $p<q$, then $m\geq m'$ and we want $B^TB=I$. From (6.1),

$$\begin{aligned}(B^TB)_{i,j}&=\sum_{n=1}^{m}\frac{f(q,p,i)f(p,q,n)^2f(q,p,j)}{((\cos\theta_n-\cos\theta'_i)(\cos\theta_n-\cos\theta'_j))}\\ &=\frac{f(q,p,i)f(q,p,j)}{\cos\theta'_j-\cos\theta'_i}\sum_{n=1}^{m}\left\{\frac{f(p,q,n)^2}{\cos\theta'_i-\cos\theta_n}-\frac{f(p,q,n)^2}{\cos\theta'_j-\cos\theta_n}\right\}\end{aligned} \tag{6.3}$$

for $i\neq j$.

We want the RHS of (6.3) to vanish for $i \neq j$. Consider the functions

$$\tilde{P}_p(c) = \prod_{i=1}^{m} (c - \cos\theta_i) = N^{-L}(c+1)^m P\left(\frac{c-1}{c+1}\right), \tag{6.4}$$

$$\mathcal{F}(c) = \sum_{n=1}^{m} \frac{f(p,q,n)^2}{c - \cos\theta_n}. \tag{6.5}$$

The first is a known function, given by (2.17) and (2.18), the second is of the form $\mathcal{R}_p(c)/\tilde{P}_p(c)$, $\mathcal{R}_p(c)$ being a polynomial of degree $m-1$. We want there to exist constants γ, γ' (dependent on p, q) such that

$$\mathcal{F}(c) = \gamma' + \gamma \tilde{P}_q(c)/\tilde{P}_p(c), \tag{6.6}$$

since then $\mathcal{F}(\cos\theta'_i) = \mathcal{F}(\cos\theta'_j) = \gamma'$ and the RHS of (6.3) vanishes. This implies that

$$\mathcal{R}_p(c) = \gamma' \tilde{P}_p(c) + \gamma \tilde{P}_q(c). \tag{6.7}$$

From (2.16), m and $m' = m_q$ differ by at most one, so $m'+1 \geq m \geq m'$. Whether $m = m'$ or $m = m'+1$, we can always choose γ' to ensure that the RHS of (6.7) is a polynomial of degree $m-1$. Then the equation defines $\mathcal{R}_p(c)$ (to within the factor γ) and the parameters $f(p,q,n)$.

From (6.5), $f(p,q,n)^2$ is the residue of $\mathcal{F}(c)$ at the pole $c = \cos\theta_n$, so from (6.6)

$$f(p,q,n)^2 = \gamma \tilde{P}_q(\cos\theta_n)/\Delta_p(\cos\theta_n), \tag{6.8}$$

where

$$\Delta_p(c) = \frac{d}{dc}\tilde{P}_p(c). \tag{6.9}$$

For given p, q, this determines $f(p,q,i)$ to within a factor independent of i (but possibly dependent on p and q). To determine this factor we need to consider the case when $i = j$ in the first of (6.3), which gives

$$f(p,q,i)^2 G(\cos\theta'_i) = 1, \tag{6.10}$$

where

$$G(c) = \sum_{n=1}^{m} f(p,q,n)^2/(c - \cos\theta_n)^2. \tag{6.11}$$

From the equations above,

$$G(c) = -\frac{d}{dc}F(c) = -\gamma \frac{d}{dc}\frac{\tilde{P}_q(c)}{\tilde{P}_p(c)}. \tag{6.12}$$

Since $\tilde{P}_q(\cos\theta'_i) = 0$, this gives

$$f(q,p,i)^2 = \frac{1}{G(\cos\theta'_i)} = -\frac{\tilde{P}_p(\cos\theta'_i)}{\gamma \Delta_q(\cos\theta'_i)}. \tag{6.13}$$

The parameter γ is at our disposal. We observe numerically that for small values of n and L we can ensure that $f(p,q,n)^2$, $f(q,p,i)^2$ are real and positive by choosing

$$\gamma = 1. \tag{6.14}$$

We can then take $f(p,q,n)$, $f(q,p,i)$ to be positive, for all n,i. The matrix B is then defined by (6.1), (6.8), (6.13), (6.14). It is real and has the orthogonality property $B^T B = I$. If $m = m'$ this implies $BB^T = I$.

6.2 The Case $p > q$

We can combine (6.8), (6.13) into a single formula by defining

$$\epsilon(p,q) = 1 \quad \text{if } p < q, \qquad \epsilon(p,q) = -1 \quad \text{if } p > q. \tag{6.15}$$

Then both equations are contained in

$$f(p,q,i) = \left[\epsilon(p,q)\tilde{P}_q(\cos\theta_i)/\Delta_p(\cos\theta_i)\right]^{1/2}, \tag{6.16}$$

for $p \neq q$.

We can now extend the formula (6.1) to all $p \neq q$. It is readily observed that

$$B_{q,p} = -B^T_{p,q}. \tag{6.17}$$

We have just established that $B^T B = I$ if $p < q$. It follows that $BB^T = I$ if $p > q$ (which implies $m \le m'$). This is the desired orthogonality property.

We remark that we have only conjectured (based on numerical calculations) that the RHS of (6.16) is real and can be chosen positive. If this were to fail the above formulae would still apply, but B_{pq} would be a complex orthogonal matrix.

6.3 The Matrix E

We shall also need the m by m diagonal matrix E_{pq}, with entries

$$[E_{p,q}]_{i,j} = e(p,q,i)\,\delta_{i,j}, \tag{6.18}$$

where the function $e(p,q,i)$ is defined as follows, for $0 \le p,q < N$:

$$\begin{aligned} e(p,q,i) &= \sin\theta_i && \text{if } p < q \text{ and } m > m' \\ &= \tan(\theta_i/2) && \text{if } p < q \text{ and } m = m' \\ &= 1/\sin\theta_i && \text{if } p > q \text{ and } m < m' \\ &= \cot(\theta_i/2) && \text{if } p > q \text{ and } m = m'. \end{aligned} \tag{6.19}$$

Since $m - 1 \le m' \le m$ if $p < q$, and $m + 1 \ge m' \ge m$ if $p > q$, these equations cover all cases; $\theta_i = \theta(p,i)$ is again as defined in (2.18). The function $e(q,p,i)$ is defined similarly, but with p,q interchanged and θ_i replaced by $\theta'_i = \theta(q,i)$.

7 The Conjecture for *W*

We return to considering the $\widetilde{W}_{p,q}$ of (5.23), (5.33) and (5.37). Based on the calculation for the Ising model [3, (7.9)], we conjecture that

$$\widetilde{W}_{p,q}(\alpha,\beta,x) = \tilde{Z}_p(\alpha)\tilde{Z}_q(\beta)\mathcal{D}_{p,q}(\alpha,\beta), \tag{7.1}$$

where $\mathcal{D}_{p,q}(\alpha,\beta)$ is the m by m determinant

$$\mathcal{D}_{p,q}(\alpha,\beta) = \det[I_m - xX_p(\alpha)E_{p,q}B_{p,q}X_q(\beta)E_{q,p}B_{q,p}] \tag{7.2}$$

or equivalently the m' by m' determinant

$$\mathcal{D}_{p,q}(\alpha,\beta) = \det[I_{m'} - xX_q(\beta)E_{q,p}B_{q,p}X_p(\alpha)E_{p,q}B_{p,q}]. \tag{7.3}$$

Again I_m is the identity matrix, of dimension m and $X_p(\alpha)$ is the diagonal m by m matrix whose entry in position (i,j) is

$$[X_p(\alpha)]_{i,j} = \frac{v_p(\alpha,\theta_j)}{u_p(\alpha,\theta_j)}\delta_{i,j}. \tag{7.4}$$

Note from (6.1) that each function $f(p,q,i)$, $f(q,p,j)$ occurs twice (i.e. as its square) in (7.2) and (7.3), so the choice of the square roots in (6.16) is in fact irrelevant.

From (3.17) and (5.23), $v_p(\alpha,\theta) = 0$ and $\tilde{Z}_p(\alpha) = 1$ if $\alpha = 0$, so (7.1) does indeed have the properties (5.27), (5.28). It is a fairly immediate generalization of (7.7) of [3] and has been tested to high numerical accuracy (60 digits or more) for arbitrary k', α, β and all N, L, p, q such that $2 \le N$, $3 \le L$, $N + L \le 10$. We *conjecture* that it is true for all $N, L, p, q, x, \alpha, \beta$.

7.1 Consequences

Define

$$a_{p,j} = \{1 - k'\mathrm{e}^{\mathrm{i}\theta_j}\}^{1/2}, \qquad b_{p,j} = \{1 - k'\mathrm{e}^{-\mathrm{i}\theta_j}\}^{1/2}, \tag{7.5}$$

where $\theta_1, \ldots, \theta_m$ are given by (2.18). They depend on p. Again the function m_p is defined by (2.16) for $0 \le p < N$, and $m = m_p$, $m' = m_q$.

Then from (3.16),

$$\lambda_j = \lambda(\theta_j) = (1 - 2k'\cos\theta_j + k'^2)^{1/2} = a_{p,j}b_{p,j},$$

so from (5.31) and (3.17),

$$\lim_{\alpha\to\infty} \frac{\tilde{Z}_p(\alpha)^2}{\tilde{Z}_p(2\alpha)} = \prod_{j=1}^{m} \frac{(a_{p,j} + b_{p,j})^2}{4a_{p,j}b_{p,j}}. \tag{7.6}$$

Also define quantities $x_{p,j}$, not to be confused with the x of (5.25), by

$$x_{p,j} = \lim_{\alpha\to\infty} \frac{v_p(\alpha,\theta_j)}{u_p(\alpha,\theta_j)} = \frac{-k'\sin\theta_j}{\lambda_j + 1 - k'\cos\theta_j}. \tag{7.7}$$

Then

$$x_{p,j} = \mathrm{i}\,\frac{b_{p,j} - a_{p,j}}{b_{p,j} + a_{p,j}} \tag{7.8}$$

and

$$\lim_{\alpha\to\infty} \frac{\tilde{Z}_p(2\alpha)}{\tilde{Z}_p(\alpha)^2} = \prod_{j=1}^{m} \left(1 + x_{p,j}^2\right). \tag{7.9}$$

Let

$$X_p = \lim_{\alpha\to\infty} X_p(\alpha), \tag{7.10}$$

so it is the diagonal matrix with diagonal elements $x_{p,j}$. Taking the limits $\alpha, \beta \to +\infty$ and setting $x = 1$, it follows from (5.30), (7.1) that if $q = p + r$ to modulo N, then

$$\mathcal{M}_r = \lim_{L\to\infty} \frac{\det(I_m - X_p E_{p,q} B_{p,q} X_q E_{q,p} B_{q,p})}{\{\det(I_m + X_p^2)\det(I_{m'} + X_q^2)\}^{1/2}} \tag{7.11}$$

where $0 \le p, q < N$ and $0 < r < N$.

We have not been able to evaluate the RHS of (7.11) analytically. Even for the $N = 2$ Ising case discussed in [3], we do not directly evaluate (7.11), but rather the expression in terms of square roots of L by L determinants that leads in that case to (7.11).[1]

We have conducted numerical experiments for various values of N, p, q and k', and observed that as $L \to \infty$ the expression on the RHS of (7.11) does indeed approach the known result (5.5), the error for finite L being of the order of k'^L or smaller.

8 Summary

We have defined the Hamiltonian partition functions $\widetilde{W}_{p,q}(\alpha, \beta, x)$, $\tilde{Z}_p(\alpha)$ by (5.23), (5.24) and shown that the spontaneous magnetization $\mathcal{M}_r$ of the superintegrable chiral Potts model is given by (5.30). For the general solvable chiral Potts model, $\mathcal{M}_r$ is independent of the rapidities [7, p. 7]. The superintegrable model is obtained from the general by a special choice of the rapidities (k' being the same) [1, p. 5], so $\mathcal{M}_r$ is the same for both.[2]

By taking the Hamiltonian limit of the results of [1], we show that $\tilde{Z}_p(\alpha)$ is given by (5.38). We then conjecture that $\widetilde{W}_{p,q}(\alpha, \beta, x)$ is given in terms an m by m determinant by (7.1). This is a natural generalization of the known result for the special case $N = 2$, i.e. the Ising model [3].

If this is true (and all the numerical evidence suggests that it is) this is a huge simplification, reducing the problem from exponential complexity to comparatively small polynomial complexity. Even so, we have not been able to make the final step and to obtain $\mathcal{M}_r$ from (7.11). We already know [6, 7] that $\mathcal{M}_r$ is given by (5.5), but it would be interesting to obtain it by this more algebraic route. The matrices B_{pq} and (for $m \ge m'$) $\mathcal{D}_{p,q}(\alpha, \beta) B_{p,q}$ are Pick matrices [15].

So there remain two things to do: to prove the conjecture (7.1) and to evaluate the limit (7.11). The first is an algebraic problem, the second an analytic one. The fact that (7.1) contains the additional parameters α, β, x should be helpful in establishing it.

[1] We do this by writing $\mathcal{M}_r^2$ as the determinant of a Toeplitz matrix and using Szegő's theorem.

[2] Note that the p, q of this paper are *not* rapidities.

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References

1. Baxter, R.J.: Superintegrable chiral Potts model: thermodynamic properties, an "inverse" model, and a simple associated Hamiltonian. J. Stat. Phys. **57**, 1–39 (1989)
2. Tarasov, V.O.: Cyclic monodromy matrices for the R-matrix of the six-vertex model and the chiral Potts model with fixed spin boundary conditions. Int. J. Mod. Phys. **A7**(1B), 963–975 (1992)
3. Baxter, R.J.: Algebraic reduction of the Ising model. J. Stat. Phys. **132** (2008). doi:10.1007/s10955-008-9587-y
4. Yang, C.N.: The spontaneous magnetization of a two-dimensional Ising model. Phys. Rev. **85**, 808–816 (1952)
5. Kaufman, B.: Crystal statistics. II. Partition function evaluated by spinor analysis. Phys. Rev. **76**, 1232–1243 (1949)
6. Baxter, R.J.: Derivation of the order parameter of the chiral Potts model. Phys. Rev. Lett. **94**, 130602 (2005)
7. Baxter, R.J.: The order parameter of the chiral Potts model. J. Stat. Phys. **120**, 1–36 (2005)
8. Baxter, R.J., Perk, J.H.H., Au-Yang, H.: New solutions of the star-triangle relations for the chiral Potts model. Phys. Lett. A **128**, 138–142 (1988)
9. Baxter, R.J.: The superintegrable chiral Potts model. Phys. Lett. A **133**, 185–189 (1988)
10. Albertini, G., McCoy, B.M., Perk, J.H.H., Tang, S.: Excitation spectrum and order parameter for the integrable N-state chiral Potts model. Nucl. Phys. B **314**, 741–763 (1989)
11. Au-Yang, H., Perk, J.H.H.: Eigenvectors in the Superintegrable Model II: Ground state sector. arXiv:0803.3029 (2008)
12. Au-Yang, H., Perk, J.H.H.: Onsager's star-triangle relation: master key to integrability. Adv. Stud. Pure Math. **19**, 57–94 (1989)
13. Onsager, L.: Crystal statistics. I. A two-dimensional model with an order-disorder transition. Phys. Rev. **65**, 117–149 (1944)
14. Onsager, L.: In: Proceedings of the IUPAP Conference on Statistical Mechanics, "Discussione e osservazioni". Nuovo Cim. (Suppl.), Ser 9 **6**, 261 (1949)
15. Agler, J., McCarthy, J.E.: Pick Interpolation and Hilbert Function Spaces. Grad. Stud. in Math., vol. 44. Am. Math. Soc., Providence (2002)